

ON A GRADIENT MAXIMUM PRINCIPLE FOR SOME QUASILINEAR PARABOLIC EQUATIONS ON CONVEX DOMAINS

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ABSTRACT. We establish a spatial gradient maximum principle for classical solutions to the initial and Neumann boundary value problem of some quasilinear parabolic equations on smooth convex domains.

1. STATEMENT OF MAIN THEOREM

In this note, we study the initial and Neumann boundary value problem of a quasilinear diffusion equation with a linear reaction term:

$$(1.1) \quad \begin{cases} u_t = \operatorname{div}(\sigma(Du)) - c(t)u & \text{in } \Omega \times (0, T], \\ \partial u / \partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T], \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

Here, $\Omega \subset \mathbf{R}^n$ ($n \geq 1$) is a bounded convex domain with C^2 boundary, $T > 0$ is any fixed number, $u = u(x, t)$ is the unknown function with u_t and $Du = (u_{x_1}, \dots, u_{x_n})$ denoting its rate of change and spatial gradient respectively, \mathbf{n} is the outer unit normal on $\partial\Omega$, $u_0 \in C^2(\bar{\Omega})$ is a given initial function satisfying the compatibility condition:

$$(1.2) \quad \partial u_0 / \partial \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

$c = c(t) \in W^{1,q_0}(0, T)$ is nonnegative for some $n + 2 < q_0 < \infty$, and $\sigma: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is given by $\sigma(p) = f(|p|^2)p$ ($p \in \mathbf{R}^n$) for some function $f \in C^3([0, \infty))$ fulfilling

$$\lambda \leq f(s) + 2sf'(s) \leq \Lambda \quad \forall s \geq 0,$$

where $\Lambda \geq \lambda > 0$ are *ellipticity* constants. We easily have

$$\sigma_{p_j}^i(p) = f(|p|^2)\delta_{ij} + 2f'(|p|^2)p_i p_j \quad (i, j = 1, 2, \dots, n; p \in \mathbf{R}^n)$$

and hence the *uniform ellipticity* condition:

$$\lambda|q|^2 \leq \sum_{i,j=1}^n \sigma_{p_j}^i(p) q_i q_j \leq \Lambda|q|^2 \quad \forall p, q \in \mathbf{R}^n;$$

that is, (1.1) is a quasilinear uniformly parabolic problem with conormal boundary condition. Thus the existence, uniqueness and regularity of a classical solution u to

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(1.1) follow from the standard theory such as in [9, Theorem 13.24] under suitable Hölder regularity assumptions on u_0 and $\partial\Omega$.

The main result of this note is the following theorem.

Theorem 1.1 (Gradient Maximum Principle). *If $u \in C^{2,1}(\bar{\Omega}_T)$ is a classical solution to problem (1.1), where $\Omega_T = \Omega \times (0, T]$, then it satisfies the gradient maximum principle:*

$$(1.3) \quad \|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}.$$

Gradient estimates for parabolic equations are usually given as *a priori* estimates depending on the initial datum, domain and ellipticity constants. Our result, Theorem 1.1, gives an estimate independent of the *convex* domain and ellipticity constants. In case of the heat equation ($f \equiv 1$ and $c \equiv 0$), (1.3) was proved in [3] for $C^{3,1}$ solutions and convex C^3 domains. Theorem 1.1 extends such a result to a large class of uniformly parabolic equations for $C^{2,1}$ solutions and convex C^2 domains. It is also important to note that the convexity assumption on the domain Ω in our result cannot be dropped in general; see a counterexample in [1, Theorem 4.1]. Also, we refer the reader to [10, 11] for more extensive studies on the maximum principles in elliptic and parabolic differential equations.

Our motivation of (1.3) is in the application of its pure diffusion case ($c \equiv 0$) to the study of the Neumann problem of some forward-backward diffusion equations [5, 6, 7]. Although the proof of Theorem 1.1 would become much easier if u belonged to $C^{3,1}(\bar{\Omega}_T)$, the existence of such a solution u often requires the initial datum u_0 lie in $C^{3+\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$ and satisfy, in addition to (1.2), the second compatibility condition:

$$(1.4) \quad \partial(\operatorname{div}(\sigma(Du_0)))/\partial\mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

These requirements give rise to a subtle but critical issue on the application of the convex integration method for constructing infinitely many Lipschitz solutions to certain forward-backward parabolic Neumann problems. For example, dealing with Perona-Malik type equations in [5], condition (1.4) was posted for nonconstant radial initial data $u_0 \in C^{3+\alpha}(\bar{\Omega})$ when Ω is a ball. Also, an earlier version of the main existence theorem in [6] for the Perona-Malik equation assumed that initial data $u_0 \in C^{3+\alpha}(\bar{\Omega})$ with compatibility conditions (1.2) and (1.4) satisfy some technical restrictions, which cannot handle the cases with $\|Du_0\|_{L^\infty(\partial\Omega)} \geq 1$ or with $0 < \|Du_0\|_{L^\infty(\partial\Omega)} < 1$ and $\|Du_0\|_{L^\infty(\Omega)} \geq \|Du_0\|_{L^\infty(\partial\Omega)}^{-1}$. Our main result of this note removes these requirements and restrictions on nonconstant initial data u_0 : the only requirement is that initial data $u_0 \in C^{2+\alpha}(\bar{\Omega})$ fulfill (1.2).

Another purpose of studying (1.3) (when $c \equiv 0$) is to confirm the validity of [4, Theorem 6.1] for *convex* domains. It has been a general belief that the initial-Neumann boundary value problem of a forward-backward parabolic equation in [4] admits a unique global *classical* solution if the initial datum $u_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfies (1.2) and $\|Du_0\|_{L^\infty(\Omega)} < s_0$, where $s_0 > 0$ is the threshold at which the forward parabolicity of governing equation turns into the backward one. Regarding this, many authors often reported that such a problem is well-posed for *subcritical* (or

subsonic) initial data. However, the proof of [4, Theorem 6.1] on such a result should be based on the gradient maximum principle (1.3) for a modified uniformly parabolic problem of type (1.1), and so the convexity of the domain Ω should not be overlooked in the proof as pointed out above.

We finish this section with some comments on notations. We mainly follow the notations in the book [8] for function spaces, with one exception that the letter C is used instead of H regarding suitable (parabolic) Hölder spaces. For integers $k, l \geq 0$ with $2l \leq k$, we denote by $C^{k,l}(\bar{\Omega}_T)$ [resp. $C^{k,l}(\Omega_T)$] the space of functions $u \in C^0(\bar{\Omega}_T)$ [$u \in C^0(\Omega_T)$] such that $D_x^a D_t^j u \in C^0(\bar{\Omega}_T)$ [$D_x^a D_t^j u \in C^0(\Omega_T)$] for all multiindices $|a| \leq k$ and integers $0 \leq j \leq l$ with $|a| + 2j \leq k$. We also adopt the summation convention that repeated indices in a term represent the sum.

2. PROOF OF MAIN THEOREM

We follow the notations and assumptions of Theorem 1.1 and introduce two useful lemmas. The convexity assumption on the domain Ω enters into the result (1.3) through the following lemma from [1, Lemma 2.1] or from [3, Theorem 2]; we do not reproduce the proof here.

Lemma 2.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex domain with $\partial\Omega$ of class C^2 . If $w \in C^2(\bar{\Omega})$ satisfies $\partial w / \partial \mathbf{n} = 0$ on $\partial\Omega$, then $\partial(|Dw|^2) / \partial \mathbf{n} \leq 0$ on $\partial\Omega$.*

The next lemma gives an improved interior regularity of the solution $u \in C^{2,1}(\bar{\Omega}_T)$ to problem (1.1) that enables us to apply classical Hopf's Lemma for parabolic equations in a suitable setup. Its proof is postponed until the end of this section.

Lemma 2.2. *One has*

$$(2.1) \quad u \in C^{3+\beta_0, \frac{3+\beta_0}{2}}(\Omega_T)$$

for some $0 < \beta_0 < 1$.

We now prove Theorem 1.1 based on the two lemmas above.

Proof of Theorem 1.1. Let $v = |Du|^2$ on $\bar{\Omega}_T$. By Lemma 2.2, $v \in C^{1,0}(\bar{\Omega}_T) \cap C^{2,1}(\Omega_T)$. We compute, within Ω_T ,

$$\begin{aligned} \Delta v &= 2Du \cdot D(\Delta u) + 2|D^2u|^2, \\ u_t &= \operatorname{div}(f(v)Du) - cu = f'(v)Dv \cdot Du + f(v)\Delta u - cu, \\ Du_t &= f''(v)(Dv \cdot Du)Du + f'(v)(D^2u)Du \\ &\quad + f'(v)(D^2v)Du + f'(v)(\Delta u)Dv + f(v)D(\Delta u) - cDu. \end{aligned}$$

From these equations, using $v_t = 2Du \cdot Du_t$, we obtain

$$(2.2) \quad v_t - \mathcal{L}(v) - V \cdot Dv = -2f(v)|D^2u|^2 - 2c|Du|^2 \leq 0 \quad \text{in } \Omega_T,$$

where $\mathcal{L}(v)$ and V are defined by

$$\begin{aligned} \mathcal{L}(v) &= f(|Du|^2)\Delta v + 2f'(|Du|^2)Du \cdot (D^2v)Du, \\ V &= 2f''(v)(Dv \cdot Du)Du + 2f'(v)(D^2u)Du + 2f'(v)(\Delta u)Du. \end{aligned}$$

Set $\mathcal{L}(v) = a_{ij}v_{x_i x_j}$ with coefficients $a_{ij} = a_{ij}(x, t)$, given by

$$a_{ij} = \sigma_{p_j}^i(Du) = f(|Du|^2)\delta_{ij} + 2f'(|Du|^2)u_{x_i}u_{x_j} \quad (i, j = 1, \dots, n).$$

Then, on $\bar{\Omega}_T$, all eigenvalues of the matrix (a_{ij}) lie in $[\lambda, \Lambda]$.

We now show

$$\max_{(x,t) \in \bar{\Omega}_T} v(x, t) = \max_{x \in \bar{\Omega}} v(x, 0),$$

which completes the proof. We argue by contradiction; suppose

$$(2.3) \quad M := \max_{(x,t) \in \bar{\Omega}_T} v(x, t) > \max_{x \in \bar{\Omega}} v(x, 0).$$

Let $(x_0, t_0) \in \bar{\Omega}_T$ with $v(x_0, t_0) = M$; then $t_0 > 0$. If $x_0 \in \Omega$, then the strong maximum principle [2] applied to (2.2) would imply that v is constant on $\bar{\Omega} \times [0, t_0]$, which yields $v(x, 0) \equiv M$ on $\bar{\Omega}$, a contradiction to (2.3). Consequently, $x_0 \in \partial\Omega$ and thus $v(x_0, t_0) = M > v(x, t)$ for all $(x, t) \in \Omega_T$. We can then apply Hopf's Lemma for parabolic equations [10] to (2.2) to deduce $\partial v(x_0, t_0)/\partial \mathbf{n} > 0$, which contradicts the conclusion of Lemma 2.1. \square

We finally give the proof of Lemma 2.2, although it may be well known to the experts in regularity theory.

Proof of Lemma 2.2. We rely on [8, Theorem III.12.1] for the bootstrap of interior regularity for the solution $u \in C^{2,1}(\bar{\Omega}_T)$ to problem (1.1). We divide the proof into several steps.

1. In Ω_T ,

$$(2.4) \quad u_t = \operatorname{div}(f(|Du|^2)Du) - cu = a_{ij}u_{x_i x_j} - cu,$$

where $a_{ij} = \sigma_{p_j}^i(Du) = f(|Du|^2)\delta_{ij} + 2f'(|Du|^2)u_{x_i}u_{x_j} \in C^{1,0}(\bar{\Omega}_T)$ and $c \in W^{1,q_0}(0, T) \subset C^{\alpha_0}([0, T])$ with $n + 2 < q_0 < \infty$ and $\alpha_0 := 1 - 1/q_0$. Note that the uniform ellipticity holds:

$$(2.5) \quad \lambda|\xi|^2 \leq a_{ij}(x, t)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall (x, t) \in \bar{\Omega}_T, \quad \forall \xi \in \mathbf{R}^n.$$

2. Fix an index $k \in \{1, \dots, n\}$, and set $v = u_{x_k} \in C^{1,0}(\bar{\Omega}_T)$. Differentiating (2.4) *formally* with respect to x_k , we have

$$(2.6) \quad v_t - \frac{\partial}{\partial x_j}(a_{ij}v_{x_i}) + b_i v_{x_i} + cv = g,$$

where

$$(2.7) \quad b_i = (a_{ij})_{x_j}, \quad g = (a_{ij})_{x_k}u_{x_i x_j} \in C^0(\bar{\Omega}_T), \quad c \in C^{\alpha_0}([0, T]).$$

The membership (2.7) easily verifies the admissible criteria (1.3)–(1.6) in Chapter III of [8] for coefficients and free term of equation (2.6). It is also easy to see that $v \in V_2^{1,0}(\Omega_T)$ is a weak (or generalized) solution to (2.6) in the sense of [8].

To check some additional conditions in [8, Theorem III.12.1], we rewrite equation (2.6) in non-divergence form:

$$(2.8) \quad v_t - a_{ij}v_{x_i x_j} + cv = g.$$

Choose any $n + 2 < q < \infty$. From $a_{ij} \in C^{1,0}(\bar{\Omega}_T)$ and (2.7), it follows that a_{ij} 's are bounded and continuous in Ω_T , that $\|c\|_{L^q(\Omega \times (t, t+\tau))} \rightarrow 0$ as $\tau \rightarrow 0$ for each $t \in (0, T)$, and that $g \in L^q(\Omega_T)$; that is, coefficients and free term of equation (2.8) fulfill the conditions in [8, Theorem IV.9.1] associated to the chosen number q .

With (2.5), we can now apply [8, Theorem III.12.1] to obtain that weak derivatives $v_t, v_{x_i x_j}$ ($i, j = 1, \dots, n$) exist and belong to $L^q(Q)$ for all $1 \leq q < \infty$ and domains $Q \subset \Omega_T$ with $\text{dist}(Q, \Gamma_T) > 0$, where $\Gamma_T = \bar{\Omega}_T \setminus \Omega_T$ is the parabolic boundary of Ω_T .

3. Fix any $\epsilon > 0$ sufficiently small, and let

$$\Omega^\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}, \quad \Omega_T^\epsilon = \Omega^\epsilon \times (\epsilon, T].$$

Also, fix any two indices $k, l \in \{1, \dots, n\}$, and set $w = u_{x_k x_l} \in C^0(\bar{\Omega}_T^\epsilon)$. Then by Step 2, we have $w \in V_2^{1,0}(\Omega_T^\epsilon)$. Taking *formal* derivative of (2.6) in terms of x_l , we have

$$(2.9) \quad w_t - \frac{\partial}{\partial x_j} (a_{ij} w_{x_i}) + b_i w_{x_i} + cw = h,$$

where

$$h = (a_{ij})_{x_k x_l} u_{x_i x_j} + (a_{ij})_{x_k} u_{x_i x_j x_l} + (a_{ij})_{x_l} u_{x_i x_j x_k}.$$

Since $f \in C^3([0, \infty))$, Step 2 implies

$$(2.10) \quad h \in L^q(\Omega_T^\epsilon) \quad \forall 1 \leq q < \infty.$$

Observe that coefficients of equation (2.9) are the same as those of equation (2.6). Thus as in Step 2, with (2.10), we see that the admissible criteria (1.3)–(1.6) in Chapter III of [8] are satisfied by coefficients and free term of equation (2.9). Also, $w \in V_2^{1,0}(\Omega_T^\epsilon)$ is a weak solution to (2.9).

As in Step 2, we also rewrite equation (2.9) in non-divergence form:

$$(2.11) \quad w_t - a_{ij} w_{x_i x_j} + cw = h.$$

Likewise, coefficients of (2.11) are equal to those of (2.8), and free term h satisfies (2.10).

Again with (2.5), it follows from [8, Theorem III.12.1] that weak derivatives $w_t, w_{x_i x_j}$ ($i, j = 1, \dots, n$) exist and belong to $L^q(Q)$ for all $1 \leq q < \infty$ and domains $Q \subset \Omega_T^\epsilon$ with $\text{dist}(Q, \Gamma_T^\epsilon) > 0$, where $\Gamma_T^\epsilon = \bar{\Omega}_T^\epsilon \setminus \Omega_T^\epsilon$ is the parabolic boundary of Ω_T^ϵ .

4. Set $\tilde{w} = u_t \in C^0(\bar{\Omega}_T^\epsilon)$. By Step 2, we have $\tilde{w} \in V_2^{1,0}(\Omega_T^\epsilon)$. Differentiating (2.4) *formally* with respect to t ,

$$\tilde{w}_t - \frac{\partial}{\partial x_j} (a_{ij} \tilde{w}_{x_i}) + b_i \tilde{w}_{x_i} + c\tilde{w} = \tilde{h},$$

where

$$\tilde{h} = (a_{ij})_t u_{x_i x_j} - c' u.$$

From Step 2 and $c' \in L^{q_0}(0, T)$, we have

$$\tilde{h} \in L^{q_0}(\Omega_T^\epsilon).$$

As above, we obtain from [8, Theorem III.12.1] that $\tilde{w}_t = u_{tt}$ exists and belongs to $L^{q_0}(Q)$ for all domains $Q \subset \Omega_T^\epsilon$ with $\text{dist}(Q, \Gamma_T^\epsilon) > 0$.

5. By Steps 2–4, we conclude that

$$u \in W_{q_0}^{4,2}(\Omega_T^{2\epsilon}) \quad \forall \epsilon > 0.$$

By the parabolic Sobolev embedding theorem [8, Lemma II.3.3], we obtain

$$u \in C^{3+\beta_0, \frac{3+\beta_0}{2}}(\Omega_T),$$

where $0 < \beta_0 < 1 - \frac{n+2}{q_0}$; hence (2.1) holds. \square

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